A Simple Numerical Solution for Heat Conduction in a Solid with a Receding Surface

J. J. Brogan*
Lockheed Missiles and Space Company, Sunnyvale, Calif.

Nomenclature

C_p = specific heat of the solid
H = heat rate
k = thermal conductivity
L = heat of fusion
l = thickness of slab
Q = heat absorbed
s = position of the receding surface
t = temperature
V = steady-state rate of melting from Ref. 2
x = coordinate normal to the surface
ρ = density
θ = time
ξ = time-dependent heat conduction path = $j\Delta x - s$

Subscripts

1, 2, 3 = specific subvolumes or time steps
m = melt condition
n = internal subvolume
j = index symbol
o = conditions prior to initiation of heating

Superscript

$(\ )'$ = condition encountered when dropping a nodal point

Introduction

Recent interest in use of sublimation for cooling infrared detectors to cryogenic temperatures,[1] and continued use of the ablation concept for thermal protection of re-entry vehicles, requires an accurate yet simple method of solution to problems of heat conduction in solids with a receding surface. General analytical solutions are not available and exact solutions are known only for the special cases.[2, 3] A simple numerical method of solution is described, and results of its application are presented in this note. The advantages of the present numerical method over others[2, 4–6] are the simplicity of its formulation and the ease of computation. Use of the present method permits 1) versatility in the selection of boundary conditions, e.g., a time-dependent moving boundary temperature can be incorporated easily and 2) computation without necessarily resorting to digital or analog computers. The work originally appeared in Ref. 7.

Analysis

The method is illustrated by considering one-dimensional heat conduction in a slab of thickness l insulated on the rear face and with thermal properties independent of temperature. One surface has been exposed to a heat flux H sufficient in magnitude to induce a constant melt temperature t_m. Upon melting, the molten material is instantaneously removed from the surface. The heat conduction equation and the initial and boundary conditions are as follows:

$$\rho C_p \frac{\partial t}{\partial \theta} = k \frac{\partial^2 t}{\partial x^2} \qquad s < x < l \qquad \theta > \theta_m \tag{1}$$

$$\left.\begin{aligned} t &= f(x) & x &> 0 \\ t &= t_m & x &= 0 \\ s &= 0 & & \end{aligned}\right\} \qquad \theta = \theta_m \tag{2}$$

Received June 22, 1964; revision received August 20, 1964. This work was sponsored by the Lockheed General Research Program.

* Head, Cryogenics Analysis, Large Space Vehicle Programs.

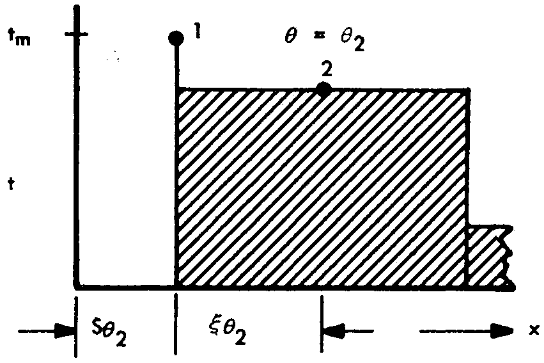

Fig. 1 Boundary location at time θ_2.

$$k \frac{\partial t}{\partial x} = 0 \qquad x = l \qquad \theta > \theta_m \tag{3}$$

$$\left.\begin{aligned} t &= t_m & x &= s \\ H &= L\rho \frac{ds}{d\theta} - k \frac{\partial t}{\partial x} & x &= s \end{aligned}\right\} \qquad \theta > \theta_m \tag{4}$$

For computational purposes, the slab is divided into subvolumes with internal subvolumes of equal thickness Δx and boundary subvolumes of thickness $\Delta x/2$. Nodal points are centrally located in each internal subvolume, and the first point is placed at the outer surface. The mean temperature of a subvolume is represented by its nodal point temperature.

As a result of this lumping process, the mean temperature of the outermost subvolume corresponds to the melt temperature. The difference formulations of the heat conduction equation for the nth internal subvolume, together with the initial and heated surface boundary conditions, are as follows:

$$\rho C_p \Delta x \frac{(t_{n,\theta+\Delta\theta} - t_{n,\theta})}{\Delta\theta} = \frac{k}{\Delta x}(t_{n-1,\theta} - t_{n,\theta}) - \frac{k}{\Delta x} \times (t_{n,\theta} - t_{n+1,\theta}) \qquad s_\theta < x < l,\ \theta \geq \theta_m \tag{5}$$

$$\left.\begin{aligned} t_1 &= t_m & x &= 0 \\ t_n &< t_m & x &> 0 \\ s &= 0 & & \end{aligned}\right\} \qquad \theta = \theta_m \tag{6}$$

$$t_1 = t_m \qquad x = s_\theta \qquad \theta > \theta_m \tag{7}$$

$$L\rho \frac{(s_{\theta+\Delta\theta} - s_\theta)}{\Delta\theta} = H - \frac{k}{\xi_\theta}(t_m = t_{j+1,\theta}) \qquad x = s_\theta;\quad \theta > \theta_m \tag{8}$$

where ξ_θ is defined as the heat conduction path between the

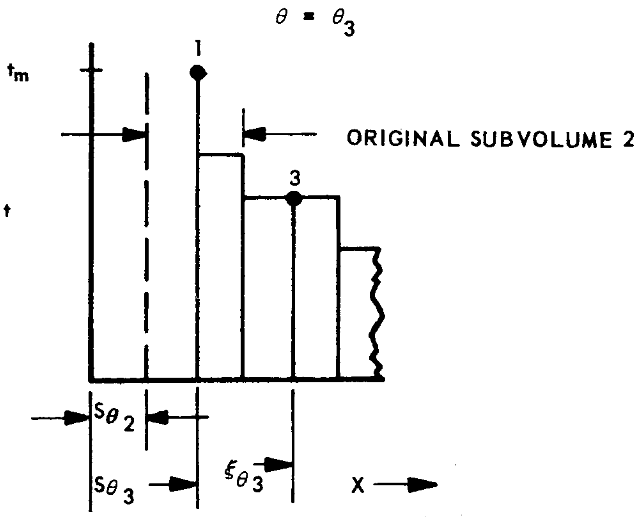

Fig. 2 Boundary location at time θ_3.

melting surface and the adjacent nodal point, or $\xi_\theta = j\Delta x - s_\theta$. During melting, nodal point 1 simulates the melting surface and moves into adjacent subvolumes and, as a result, successive nodal points are dropped from the computational grid. The subscript j is taken as one unit greater than the number of the nodal point last dropped, e.g., as the outer subvolume melts, $\xi_\theta = \Delta x - s_\theta$.

Since the entire outer subvolume is at the melt temperature, only absorption of the heat of fusion is required to initiate melting. Therefore, the first nodal point moves through the outer subvolume in discrete steps at a rate defined by Eq. (8). When the melting surface (nodal point 1) reaches the outer boundary of subvolume 2, temperature conditions are as illustrated in Fig. 1. At this particular time θ_2, the temperature of subvolume 2 is necessarily lower than the melt temperature. If melting is to continue, the exposed face of subvolume 2 must simultaneously attain the melt temperature and absorb energy. In an exact solution, the melt line proceeds into the slab by instantaneously raising the mass in its path to the melt temperature and absorbing the heat of fusion. The energy required to melt a finite thickness Δs in this subvolume is

$$Q_{\Delta s} = L\rho\Delta s + \rho C_p(t_m - t_{2,\theta_2})\Delta s \tag{9}$$

Nodal point 2 is now dropped from the network, subvolume 2 ceases to change in temperature as a unit, and a new heat of fusion is defined:

$$L' = L + c_p(t_m - t_{2,\theta_2}) \tag{10}$$

This new heat of fusion, when used in Eq. (8) in place of L, allows melting to continue into subvolume 2 and permits a mass of thickness Δs to attain, simultaneously, the melt temperature and to absorb the required heat of fusion. Melting continues in discrete steps, as illustrated in Fig. 2, with the temperature gradient at the melting surface now determined between nodal points 1 and 3. Note that the temperature of the mass between s_{θ_3} and the boundary of subvolume 3 is maintained at t_{2,θ_2}. Thus, melting continues through subvolume 2 until the melting surface reaches the outer boundary of subvolume 3 (taken at time θ_4) when a similar process is repeated, i.e., nodal point 3 is dropped from the computational grid and a new heat of fusion is defined. A similar process of dropping adjacent nodal points continues as melting proceeds. Under certain external environments the temperature of the receding surface is time-dependent. This

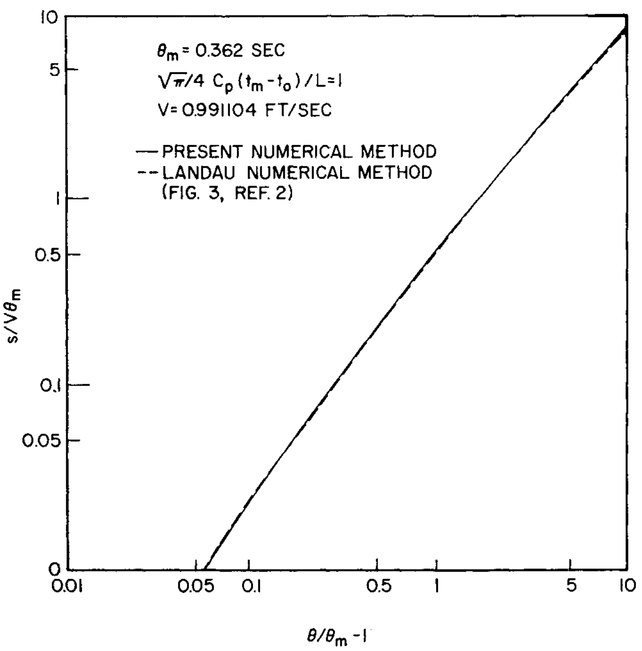

Fig. 3 Comparison of results for amount melted.

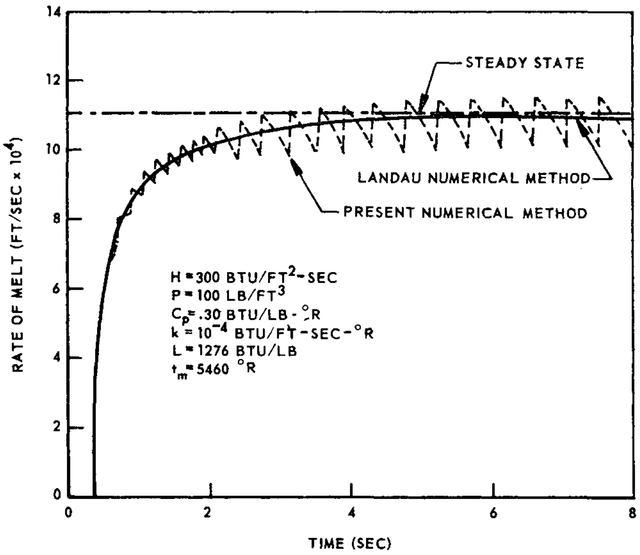

Fig. 4 Comparison of results for the rate of melt.

boundary condition can be incorporated by computing a new value of L' at each computing time step rather than one for each node. A recent work by Hurwicz et al.[6] also shows one-dimensional formulations of the moving boundary problem. These formulations require solution on an analog computer.

Comparison with Another Method

A comparison of results with a numerical and an analytical solution is presented in Figs. 3 and 4. Landau[2] presents the results of a numerical method for solution of transient melting conditions. In addition, the author derives the steady-state heat conduction solution for melting of a semi-infinite solid with exposure to a fixed heat flux. Figure 3 presents a comparison of the amount of melt in a nondimensionalized form after Landau. Crossing of these curves is a result of slight errors in the values obtained in reading from the figure in Ref. 2. However, very good agreement is seen. Figure 4 presents the comparison of melt rate. Note the oscillations occurring in the rate of melt curve for the present numerical solution. The peaks occur when a nodal point is dropped. A decrease follows as the melt line traverses a given subvolume. Also, as steady state is approached, the peaks approach the same value with time.

A comparison between the present method and the exact solution of Ref. 3 shows very good agreement in defining the temperature distribution and the amount of melt. This is reported in Ref. 7, where the convergence of the present numerical method is also shown.

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